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On the Superdiffusive Behavior of Passive Tracer with a Gaussian Drift

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In the present article we consider a motion of a passive tracer particle, whose trajectory satisfies the Itô stochastic differential equation $d\mathbf{x}(t)=\mathbf{V}(t,\mathbf{x}(t))\,dt+\sqrt{2\kappa}\,d\mathbf{w}(t)$, where $\mathbf{w}(\cdot)$ is a Brownian motion, $\mathbf{V}$ is a stationary Gaussian random field with incompressible realizations independent of $\mathbf{w}(\cdot)$ and $\kappa>0$. We prove the superdiffusive character of the motion under certain conditions on the energy spectrum of the velocity field. The result is shown both for steady (time independent) and time dependent and Markovian velocity fields. In addition, we provide explicit upper and lower bounds for the Hurst exponent of the trajectory. All previous rigorous results concerned explicitely solvable shear flows cases.

KEY WORDS: Passive tracer; superdiffusion; turbulent flow.

1. INTRODUCTION

A very simple model of a passive tracer motion in a turbulent medium is provided by an Itô stochastic differential equation equation

$$\begin{cases} d\mathbf{x}(t)=\mathbf{V}(t,\mathbf{x}(t))\,dt+\sqrt{2\kappa}\,d\mathbf{w}(t), & t\geqslant 0,\\ \mathbf{x}(0)=\mathbf{0} \end{cases}\tag{1.1}$$

Here $\mathbf{V}(\cdot,\cdot)$ is a time-space stationary, zero mean, d-dimensional random field with incompressible realizations given over a probability space $\mathcal{T}_0:=(\Omega,\mathcal{V},\mu)$, $\mathbf{w}(\cdot)$ is a d-dimensional standard Brownian motion given over

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$\mathscr{T}_1 := (\Sigma, \mathscr{W}, Q)$. The parameter κ is sometimes referred to as the *molecular diffusivity* and throughout this paper we assume that it is strictly positive. The trajectory process $\mathbf{x}(\cdot) = (x_1(\cdot),..., x_d(\cdot))$ is considered over the product probability space $\mathscr{T}_0 \otimes \mathscr{T}_1 := (\Omega \times \Sigma, \mathscr{V} \otimes \mathscr{W}, \mu \otimes Q)$. Let $\langle \cdot \rangle$, $\mathbb{E}$ denote the expectation operators corresponding to μ and $\mu \otimes Q$ respectively. Let also

$$\mathbf{R}(t, \mathbf{x}) := \langle \mathbf{V}(t, \mathbf{x}) \otimes \mathbf{V}(0, \mathbf{0}) \rangle, \qquad (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$$

be the covariance matrix of $\mathbf{V}$.

The asymptotic behavior of $\mathbf{x}(\cdot)$ is called *diffusive* when

$$\lim_{t \uparrow +\infty} \frac{d_{i,j}(t)}{t} = d^*_{i,j}, \qquad \forall\, i, j = 1,..., d, \tag{1.2}$$

with $d_{i,j}(t) := \mathbb{E}[x_i(t)\, x_j(t)]$ and $d^*_{i,j}$ finite. It is well known, see, e.g., ref. 13, (both in the steady, i.e., time independent, and non-steady cases) that for a zero mean flow ($\langle \mathbf{V} \rangle = \mathbf{0}$) the principal condition that guarantees the diffusive behavior of trajectories is

$$\int_{\mathbb{R}^d} \frac{e(d\mathbf{k})}{|\mathbf{k}|^2} < +\infty. \tag{1.3}$$

Here the measure $e(\cdot)$, called *the energy spectrum* of the field, is defined as $e(\cdot) :=$ trace $\hat{\mathbf{R}}(\cdot)$, where

$$\mathbf{R}(0, \mathbf{x}) = \int_{\mathbb{R}^d} e^{i\mathbf{x} \cdot \mathbf{k}}\, \hat{\mathbf{R}}(d\mathbf{k}).$$

The matrix $\mathbf{D}^* = [d^*_{i,j}]$, given by (1.2), is called then the *effective diffusivity* of the medium.

On the other hand, simple *shear layer flow* examples show that, when (1.3) is not satisfied, the particle motion instead of being diffusive is rather superdiffusive. In fact in a series of articles, see, e.g., refs. 1–4, Avellaneda and Majda gave a complete characterization of the tracer motion corresponding to various families of shear layer random flows. Recently similar results had been obtained by Ben Arous and Owhadi for a superpositions of periodic shear flows taking place on an infinite number of scales, ref. 5.

In the present article we set out to investigate the superdiffusive behavior of the particle motion under a quite general family of Gaussian flows (in particular non-shear flows). We consider both steady and non-steady velocity fields.

In the first case we suppose that the field $\mathbf{V}(t, \mathbf{x}) \equiv \mathbf{V}(\mathbf{x})$ is isotropic and its energy spectrum satisfies *the power law*. More specifically we assume that the covariance matrix of the field is given by

$$\mathbf{R}(\mathbf{x}) = \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\mathbf{k}} \frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}} \Gamma(\mathbf{k})\, d\mathbf{k}, \qquad \mathbf{x} \in \mathbb{R}^d, \tag{1.4}$$

with

$$\Gamma(\mathbf{k}) := \mathbf{I} - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}.$$

Here a cut-off function $a: [0, +\infty) \to [0, +\infty)$ is continuous at 0 with $a(0) > 0$ and satisfies $\operatorname{supp} a(\cdot) \subseteq [0, K]$ for a certain $K > 0$. The matrix Γ appearing in (1.4) guarantees that the realizations of the field are divergenceless, i.e., $\nabla_{\mathbf{x}} \cdot \mathbf{V}(\mathbf{x}) \equiv 0$, μ-a.s. To ensure integrability of the spectrum we assume further that $\alpha < 1$. The parameter α is directly related to the decay exponent of $\mathbf{R}$, namely $\mathbf{R}(\mathbf{x}) \sim |\mathbf{x}|^{\alpha-1}$ for $|\mathbf{x}| \gg 1$. As α increases to one, the spatial decay exponent of $\mathbf{R}$ decreases to zero and, consequently, spatial correlation of velocity increases.

Note also that in our case (1.3) is fulfilled iff $\alpha < 0$, so in this case the limits in (1.2) exist and the behavior of the particle is diffusive. We show, in Theorem 1 later, that when $\alpha \in (0, 1)$ the behavior of the particle is super-diffusive in the following sense. There exists $\gamma_*, \underline{c} > 0$ such that

$$C-\liminf_{t\uparrow+\infty} \frac{1}{t^{1+\gamma_*}} \mathbb{E}\, |\mathbf{x}(t)|^2 \geqslant \underline{c} \tag{1.5}$$

The symbol C-lim inf denotes the lim inf of the Césaro averages.

In the non-steady case we consider a family of time dependent, stationary Gaussian Markovian fields with the covariance matrix given by

$$\mathbf{R}(t, \mathbf{x}) = \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\mathbf{k}}\, e^{-|\mathbf{k}|^{2\beta} t} \frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}} \Gamma(\mathbf{k})\, d\mathbf{k} \tag{1.6}$$

with $a(\cdot)$, α as in (1.4) and $\beta \geqslant 0$. The function $\exp(-|\mathbf{k}|^{2\beta} t)$ in (1.6) is called *the time correlation function* of the velocity $\mathbf{V}$ corresponding to the wave number $\mathbf{k}$. For $\beta = 0$ the field possesses the *spectral gap property*, cf. ref. 7, i.e., the speed of time decorrelation is uniform for all wave numbers. On the other hand, for $\beta > 0$, the velocity field lacks the spectral gap and thus strong (time) mixing property. In this case the diffusive regime is a result of a subtle balance between temporal and spatial mixing properties of the field. It can be shown, see refs. 10, 12, that for

$$\alpha < 0, \qquad \text{or} \quad \alpha \in (0, 1) \quad \text{but} \ \alpha + \beta < 1 \tag{1.7}$$

(see the region I of Fig. 1) the limits in (1.2) exist (*diffusive behavior*).

In our second result, see Theorem 2 and the accompanying it Fig. 1, we prove the super-diffusive behavior of the tracer particle, in the sense of (1.5), provided that both conditions mentioned in (1.7) are violated.

In addition, in both steady and time dependent cases we provide lower and upper estimates of the parameter γ_*. In the time dependent case this leads to distinguishing three separate regimes concerning the values of the parameters α, β appearing in the definition of the spectrum. They correspond to the regions II, III, IV of Fig. 1. In fact the estimate of γ_* in region II appears to lead to a definite value of the superdiffusivity exponent namely $\mathbb{E}\,|\mathbf{x}(t)|^2 \sim t^{(\alpha+2\beta-1)/\beta}$, $t \gg 1$. This result coincides with the heuristic argument given in ref. 10.

Theorems 1 and 2 are, according to our knowledge, one of the first results showing rigorously the superdiffusive behavior of the motion of a passive tracer displayed in a model that is not explicitly solvable. While preparing the article we have learned (ref. 9) that similar in nature results have been obtained by Owhadi for deterministic velocities that are superpositions of periodic flows taking place on an infinite number of scales.

The method we use to prove both Theorems 1 and 2 relies on the variational principles, formulated in Proposition 1 later and proved

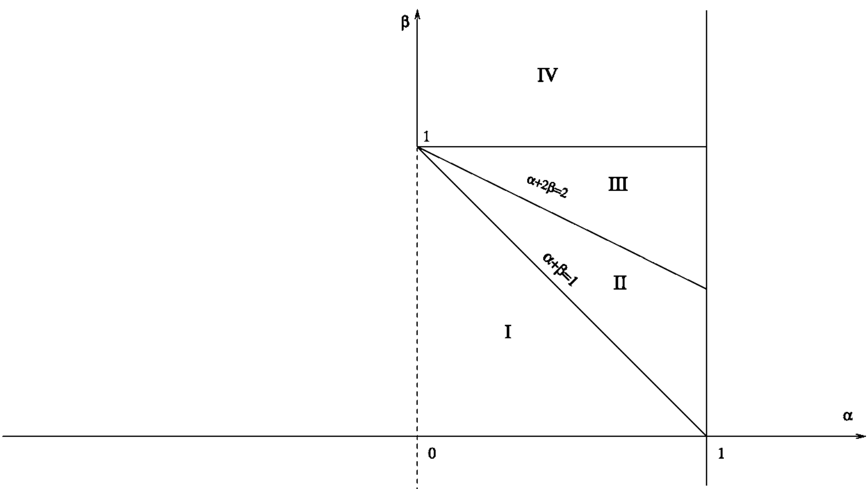

Fig. 1. The regimes corresponding to a time dependent flow.

in ref. 15. These principles allow us to estimate the Laplace transform $\int_0^{+\infty} e^{-\lambda t} \mathbf{D}(t)\, dt$, where $\mathbf{D}(t) = [d_{i,j}(t)]$, see (1.2), for small $\lambda > 0$. The test functions we use to derive those bounds are the first degree polynomials in an appropriate Gaussian Hilbert space generated by the velocity field.

Thanks to an easy upper bound (3.8) and a Tauberian type of result given in (3.9) the bounds on the Laplace transform lead to both lower and upper bounds for $\mathbf{D}(t)$ for $t \gg 1$.

We finish this section pointing out the difference between the scaling exponents appearing in our model and those of the shear layer model of refs. 1–4. It is caused by the absence of any infrared cut-off for the wave-numbers in the spectral measure considered here, see (1.6).

This fact explains why we obtain a different super-diffusivity exponent in the only region we know its precise value, i.e., region II, from the one of ref. 3.

2. THE FORMULATION OF THE MAIN RESULTS

2.1. The Time Independent Case

Let $\mathbf{V}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$ be a zero mean Gaussian homogeneous field with the covariance matrix given by (1.4). Let us introduce the following notation:

$$\gamma^* := \sup[\gamma \geqslant 0 : \liminf_{t \uparrow +\infty} t^{-1-\gamma}\, \mathbb{E}\, |\mathbf{x}(t)|^2 > 0] \tag{2.1}$$

and

$$\gamma_* := \sup\left[\gamma \geqslant 0 : \lim_{t \uparrow +\infty} t^{-2-\gamma} \int_0^t \mathbb{E}\, |\mathbf{x}(s)|^2\, ds = +\infty\right], \tag{2.2}$$

where $\mathbf{x}(\cdot)$ is given by (1.1). Obviously, $\gamma_* \leqslant \gamma^*$.

Theorem 1. Under the assumptions about the field $\mathbf{V}$ made above we have

$$\gamma_* \geqslant \alpha^2, \qquad \text{when} \quad \alpha \in (0, 1/2] \tag{2.3}$$

$$\gamma_* \geqslant \frac{\alpha}{3 - 2\alpha}, \qquad \text{when} \quad \alpha \in (1/2, 1) \tag{2.4}$$

In addition,

$$\gamma^* \leqslant \alpha, \qquad \text{when} \quad \alpha \in (0, 1) \tag{2.5}$$

Remark. The upper estimate for γ^* asserted in Theorem 1 is sharp for the shear layer model, see ref. 4.

2.2. The Time Dependent Case

In this section we assume that the field $\mathbf{V}(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ is time dependent, zero mean, Gaussian, with the covariance matrix given by (1.6).

Theorem 2. Suppose that the field satisfies the assumptions made above and γ_*, γ^* are defined by (2.2) and (2.1). Then,

(i) $\gamma_* = \gamma^* = \frac{\alpha+\beta-1}{\beta}$, when $1 < \alpha + \beta$ and $\alpha + 2\beta < 2$, see region II on Fig. 1.

(ii) $\frac{1-\beta}{2-\alpha-\beta} \leqslant \gamma_* \leqslant \gamma^* \leqslant \frac{\alpha+\beta-1}{\beta}$, when $2 < \alpha + 2\beta$, $\beta < 1$, see region III on Fig. 1.

(iii) Both γ_* and γ^* are given by (2.3), (2.5) when $\beta > 1$, see region IV on Fig. 1.

Remark 1. A heuristic argument of ref. 10 shows that in region II of Fig. 1 the laws of continuous trajectory processes $\varepsilon \mathbf{x}(t/\varepsilon^q)$, $t \geqslant 0$ with $q := \frac{2\beta}{\alpha+2\beta-1}$ converge weakly, as $\varepsilon \downarrow 0$, to a fractional Brownian motion with the Hurst exponent $H = \frac{\alpha+2\beta-1}{2\beta}$. This result has been proven rigorously for the weak fluctuation limit in ref. 9.

Remark 2. As in the time independent case a simple direct calculation shows that the upper estimates for γ^* asserted in Theorem 2 are sharp for the shear layer model. Indeed, in this case the particle motion is described by

$$\begin{cases} x(t) = \int_0^t V(s, w_2(s))\, ds + w_1(t), \\ y(t) = w_2(t), \end{cases}$$

where $w_1(\cdot)$, $w_2(\cdot)$ are two independent one dimensional standard Wiener processes and $V(\cdot, \cdot)$ is an independent of them time-space stationary Gaussian field with the covariance given by

$$R(t, y) = \langle V(t, y)\, V(0, 0) \rangle = \int_{|k| \leqslant K} e^{\mathrm{i}\, ky} e^{-|k|^{2\beta} t} \frac{dk}{|k|^{2\alpha-1}}, \tag{2.6}$$

with α, β as in (1.6) and $K > 0$ a fixed ultraviolet cut-off.

A direct calculation leads to the following formula

$$\mathbb{E} x^2(t) = 2 \int_0^t ds \int_0^s \int_{\mathbb{R}} R(u, y) \exp\left\{ -\frac{y^2}{2u} \right\} \frac{du\, dy}{\sqrt{2\pi u}} + t \tag{2.7}$$

Using (2.6) we obtain that the right hand side of (2.7) equals

$$2t\int_0^K \left[1-\frac{1-\exp\{-(k^{2\beta}+k^2)\,t\}}{(k^{2\beta}+k^2)\,t}\right]\frac{dk}{(k^{2\beta}+k^2)\,k^{2\alpha-1}}+t$$

After a straightforward calculation we conclude that $\mathbb{E}x^2(t)\sim ct$, $t\gg 1$ for some $c>0$, when $\alpha<0$, or $\alpha+\beta<1$ (cf. region I of Fig. 1), $\mathbb{E}x^2(t)\sim ct^{(\alpha+2\beta-1)/\beta}$, $t\gg 1$, when $\alpha+\beta>1$ and $\beta<1$ (cf. regions II and III of Fig. 1). Finally, $\mathbb{E}x^2(t)\sim ct^{1+\alpha}$, $t\gg 1$, when $\beta>1$ (cf. region IV of Fig. 1).

3. AUXILIARIES

3.1. Estimates of the Variance of an Additive Functional of a Markov Process with the Help of Variational Principles

Suppose that (Ω,d) is a Polish metric space with $\mathscr{V}$ its Borel σ-algebra. Let $(\eta_t)_{t\geqslant 0}$ be a certain Markov process, with Ω its state space, defined over the probability space $\mathscr{T}_0\otimes\mathscr{T}_1$. By $(Q^t)_{t\geqslant 0}$ we denote its transition of probability semigroup, i.e., the semigroup of operators defined on the space $B(\Omega)$ consisting of bounded and Borel measurable functions on Ω satisfying

$$\mathbb{E}[F(\eta_{t+h})\mid\mathscr{Z}_t]=Q^hF(\eta_t)$$

for any $t, h\geqslant 0$, $F\in B(\Omega)$. Here $(\mathscr{Z}_t)$ is the natural filtration of σ-algebras corresponding to the process and $\mathbb{E}[\,\cdot\mid\mathscr{Z}_t]$ is the respective conditional expectation operator.

Suppose further that μ is invariant under the process, i.e., $\int Q^tF\,d\mu=\int F\,d\mu$ for any $t\geqslant 0$, $F\in B(\Omega)$ and that the semigroup $(Q^t)_{t\geqslant 0}$ extends to a C_0-continuous semigroup of contractions on $L^2:=L^2(\mu)$. By $(\cdot,\cdot)_{L^2}$ we denote the respective scalar product on L^2.

Let $\mathscr{L}$ be the generator of the semigroup and let

$$R_\lambda F:=\int_0^{+\infty}\mathrm{e}^{-\lambda t}Q^tF\,dt,\qquad \lambda>0,\quad F\in L^2$$

denote the family of resolvent operators corresponding to the process.

Suppose also that the bilinear form

$$\mathscr{E}_{\mathscr{L}}(F,G):=(-\mathscr{L}F,G)_{L^2},\qquad (F,G)\in\mathscr{D}(L)\times L^2 \tag{3.1}$$

is *closable*, see ref. 16, p. 28 for the definition of closability. Define

$$\mathscr{E}^s_{\mathscr{L}}(F,G):=\tfrac{1}{2}\,[\mathscr{E}_{\mathscr{L}}(F,G)+\mathscr{E}_{\mathscr{L}}(G,F)]$$

the symmetric part of $\mathscr{E}_{\mathscr{L}}$. Thanks to the closability assumption there exists a self-adjoint operator $S: \mathscr{D}(S) \to L^2$, the *symmetric part of the generator*, such that $\mathscr{D}(\mathscr{L}) \subseteq \mathscr{D}(S^{1/2})$ and $\mathscr{E}^s_{\mathscr{L}}(F, G) = (S^{1/2}F, S^{1/2}G)_{L^2}$, $F, G \in \mathscr{D}(\mathscr{L})$, see ref. 16. Suppose further that $\mathscr{C} := \mathscr{D}(\mathscr{L}) \cap \mathscr{D}(S)$ forms a core of $\mathscr{L}$ and define the *anti-symmetric* part of $\mathscr{L}$ as $AF := \mathscr{L}F - SF$.

Let $\mathscr{H}_1$ be the Hilbert space obtained as the closure of $\mathscr{C}_0 := \mathscr{C} \cap L^2_0$ under any norm given by $\|f\|^2_{1,\lambda} := \lambda \|f\|^2_{L^2} + \|S^{1/2}f\|^2_{L^2}$ with $\lambda > 0$. Here $L^2_0 := \{f \in L^2 : \int f \, d\mu = 0\}$. We define $L^2_0 \subseteq \mathscr{H}_{-1}$ as the dual to $\mathscr{H}_1$ under $(\cdot, \cdot)_{L^2}$ pairing. According to Section 2.4, pp. 19–22 of ref. 18 its norm, restricted to L^2_0, is given by

$$\|f\|^2_{-1,\lambda} := \sup_{g \in \mathscr{C}} [2(f, g)_{L^2} - \|g\|^2_{1,\lambda}] < +\infty \tag{3.2}$$

Suppose that $V: \Omega \to \mathbb{R}$ is a random variable over $\mathscr{T}_0$, such that $V \in L^2$ and

$$Y(t) := \int_0^t V(\eta_s) \, ds, \qquad t \geqslant 0 \tag{3.3}$$

Let $\lambda > 0$ and denote $\chi_\lambda := R_\lambda V \in \mathscr{D}(\mathscr{L})$, i.e.,

$$\lambda \chi_\lambda - \mathscr{L} \chi_\lambda = V \tag{3.4}$$

Denote

$$d_\lambda(V) := \|\chi_\lambda\|^2_{1,\lambda} = (R_\lambda V, V)_{L^2} \tag{3.5}$$

The following variational principle has been proven in ref. 15, see Proposition 5.1, p. 216.

Proposition 1 (The Variational Principle). The following equalities hold

$$d_\lambda(V) = \sup_{F \in \mathscr{C}} \mathscr{S}(F) \tag{3.6}$$

where $\mathscr{S}(F) := 2(V, F)_{L^2} - \|AF\|^2_{-1,\lambda} - \|F\|^2_{1,\lambda}$ and

$$d_\lambda(V) = \inf_{F \in \mathscr{C}} \mathscr{L}(F) \tag{3.7}$$

where $\mathscr{L}(F) := \|V - AF\|^2_{-1,\lambda} + \|F\|^2_{1,\lambda}$.

To use the above variational principle in order to investigate the asymptotics of $\mathbb{E}\,|Y(t)|^2/t$ for $t \gg 1$ we shall need the following result.

Lemma 1. We have

$$\frac{1}{t}\,\mathbb{E}\,|Y(t)|^2 \leqslant 16(R_{1/t}V, V)_{L^2}, \qquad \forall t > 0. \tag{3.8}$$

In addition, if there exist $\kappa, C > 0$ such that

$$(R_\lambda V, V)_{L^2} \geqslant C\lambda^{-\kappa}, \qquad \forall \lambda \in (0, 1)$$

then for any $\kappa' \in (0, \kappa)$ there exists $C_1 > 0$ such that

$$\frac{1}{t^{2+\kappa'}} \int_0^t \mathbb{E}\,|Y(s)|^2\, ds \geqslant C_1. \tag{3.9}$$

Proof. We can rewrite the right hand side of (3.3) as

$$\lambda \int_0^t \chi_\lambda(\eta_s)\, ds + M_\lambda(t) + \chi_\lambda(\eta_0) - \chi_\lambda(\eta_t), \tag{3.10}$$

where

$$M_\lambda(t) := \chi_\lambda(\eta_t) - \chi_\lambda(\eta_0) - \int_0^t \mathscr{L}\chi_\lambda(\eta_s)\, ds$$

is a $(\mathscr{Z}_t)$-martingale, with $\mathbb{E}M_\lambda^2(t) = 2t\, \|S^{1/2}\chi_\lambda\|_{L^2}^2$. Using (3.5) we conclude that

$$\frac{1}{t}\,\mathbb{E}Y^2(t) \leqslant \frac{4}{t}\,[(\lambda t)^2\, \|\chi_\lambda\|_{L^2}^2 + t\, \|\chi_\lambda\|_1^2 + 2\, \|S^{1/2}\chi_\lambda\|_{L^2}^2]$$

and (3.8) follows upon the substitution $\lambda := 1/t$.

(3.9) follows from the following simple Tauberian type result modeled on Lemma 4.5, p. 295 of ref. 22.

Lemma 2. Suppose that $f\colon [0, +\infty) \to \mathbb{R}$ is a non-decreasing, positive function, for which there exist $\varrho, \bar{C}, \underline{c} > 0$ and $\kappa \in (0, \varrho)$ such that

(i) $f(t) \leqslant \bar{C}t^{2+\varrho}$ for all $t > 0$

(ii) $\int_0^{+\infty} e^{-\lambda t} f(t)\, dt \geqslant \underline{c}\, \lambda^{-3-\kappa}$ for all $\lambda \in (0, 1)$.

Then, for any $\kappa' \in (0, \kappa)$ there exists $\tilde{c} > 0$ such that

$$f(t) \geqslant \tilde{c}t^{2+\kappa'}, \qquad \forall t > 0. \tag{3.11}$$

Proof. Suppose that $\gamma > 0$ is arbitrary. We can write then, using (ii), that

$$\underline{c}\lambda^{-3-\kappa} \leqslant \int_0^{+\infty} \mathrm{e}^{-\lambda t} f(t)\, dt = \frac{1}{\lambda}\int_0^{+\infty} \mathrm{e}^{-t} f\left(\frac{t}{\lambda}\right) dt. \tag{3.12}$$

The utmost right hand side of (3.12) equals

$$\frac{1}{\lambda}\int_0^{\lambda^{-\gamma}} \mathrm{e}^{-t} f\left(\frac{t}{\lambda}\right) dt + \frac{1}{\lambda}\int_{\lambda^{-\gamma}}^{+\infty} \mathrm{e}^{-t} f\left(\frac{t}{\lambda}\right) dt.$$

This expression can be estimated using monotonicity of f and (i) by

$$\frac{1}{\lambda} f(\lambda^{-1-\gamma}) + \frac{\bar{c}}{\lambda}\int_{\lambda^{-\gamma}}^{+\infty} \mathrm{e}^{-t}\left(\frac{t}{\lambda}\right)^{2+\varrho} dt \leqslant \frac{1}{\lambda} f(\lambda^{-1-\gamma}) + \frac{c_1}{\lambda^{3+\varrho}}\, \mathrm{e}^{-\lambda^{-\gamma}/2}$$

for some constant $c_1 > 0$. We have shown therefore that

$$\underline{c}\lambda^{-3-\kappa} \leqslant \frac{1}{\lambda} f(\lambda^{-1-\gamma}) + \frac{c_1}{\lambda^{3+\varrho}}\, \mathrm{e}^{-\lambda^{-\gamma}/2}. \tag{3.13}$$

Moving over the second expression on the right hand side of (3.13) to the left hand side of the inequality, using the fact that

$$\frac{c_1}{\lambda^{3+\varrho}}\, \mathrm{e}^{-\lambda^{-\gamma}/2} \leqslant \tfrac{1}{2}\, \underline{c}\, \lambda^{-3-\kappa}$$

for sufficiently small $\lambda > 0$, we conclude (3.11). ▮

3.2. Homogeneous Gaussian Fields

Suppose that m is a positive integer and $\vartheta_\rho(\mathbf{x}) := (1+|\mathbf{x}|^2)^{-\rho}$, $\mathbf{x} \in \mathbb{R}^d$, where $\rho > d/2$. Let Ω be the Hilbert space of d-dimensional incompressible vector fields that is the completion of $C^\infty_{0,\,\mathrm{div}} := \{\varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d) : \nabla_{\mathbf{x}} \cdot \omega = 0\}$ with respect to the norm

$$\|\omega\|_\Omega^2 := \int_{\mathbb{R}^d} (|\omega(\mathbf{x})|^2 + |\nabla_{\mathbf{x}}\omega(\mathbf{x})|^2 + \cdots + |\nabla_{\mathbf{x}}^m \omega(\mathbf{x})|^2)\, \vartheta_\rho(\mathbf{x})\, d\mathbf{x}.$$

We shall assume that $m > d/2+1$ so any $\omega \in \Omega$ is of C^1 class of regularity.

We shall identify measure μ with the law in Ω of the Gaussian velocity field, whose covariance matrix is given by (1.4). The measure μ is therefore Gaussian of zero mean and homogeneous, i.e., $\int \omega(\mathbf{x})\, \mu(d\omega) = \mathbf{0}$ and

$\mu\tau_{\mathbf{x}} = \mu$, $\forall \mathbf{x} \in \mathbb{R}^d$. Here $\tau_{\mathbf{x}}: \Omega \to \Omega$ is given by $\tau_{\mathbf{x}}\omega(\cdot) := \omega(\mathbf{x} + \cdot)$. In what follows we shall identify the flow with $\mathbf{V}(\mathbf{x}; \omega) := \mathbf{V}(\tau_{\mathbf{x}}(\omega))$, $\mathbf{x} \in \mathbb{R}^d$, where

$$\mathbf{V}(\omega) := \omega(\mathbf{0}). \tag{3.14}$$

It is easy to see that the field defined in this way is of zero mean, Gaussian with the covariance matrix given by (1.4).

3.3. Spectral Representation of Homogeneous Fields

By the Spectral Theorem (see, e.g., ref. 21) we know that there exist two independent, identically distributed, real vector valued Gaussian spectral measures $\hat{\mathbf{V}}_l(\cdot)$, $l = 0, 1$ defined over $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ with values in L^2 such that

$$\mathbf{V}(\mathbf{x}) = \int \hat{\mathbf{V}}_0(\mathbf{x}, d\mathbf{k}), \tag{3.15}$$

where

$$\hat{\mathbf{V}}_0(\mathbf{x}, d\mathbf{k}) := c_0(\mathbf{k}\cdot\mathbf{x})\, \hat{\mathbf{V}}_0(d\mathbf{k}) + c_1(\mathbf{k}\cdot\mathbf{x})\, \hat{\mathbf{V}}_1(d\mathbf{k}),$$

with $c_0(\phi) \equiv \cos(\phi)$, $c_1(\phi) \equiv \sin(\phi)$. Define also

$$\hat{\mathbf{V}}_1(\mathbf{x}, d\mathbf{k}) := -c_1(\mathbf{k}\cdot\mathbf{x})\, \hat{\mathbf{V}}_0(d\mathbf{k}) + c_0(\mathbf{k}\cdot\mathbf{x})\, \hat{\mathbf{V}}_1(d\mathbf{k}).$$

We write these two stochastic measures component-wise

$$\hat{\mathbf{V}}_l(\mathbf{x}, d\mathbf{k}) = (\hat{V}_{l,1}(\mathbf{x}, d\mathbf{k}),\ldots, \hat{V}_{l,d}(\mathbf{x}, d\mathbf{k})), \qquad l = 0, 1.$$

The following relation holds

$$\partial\hat{\mathbf{V}}_0(\mathbf{x}, d\mathbf{k})/\partial x_j = k_j \hat{\mathbf{V}}_1(\mathbf{x}, d\mathbf{k}), \tag{3.16}$$

$$\partial\hat{\mathbf{V}}_1(\mathbf{x}, d\mathbf{k})/\partial x_j = -k_j \hat{\mathbf{V}}_0(\mathbf{x}, d\mathbf{k}), \qquad j = 1,\ldots, d. \tag{3.17}$$

One can check that $\int \hat{\mathbf{V}}_1(\mathbf{x}, d\mathbf{k})$ is a random field distributed identically to and independently of $\mathbf{V}$.

For a given function ψ from a certain admissible class $\mathfrak{C}$, see (3.20) later for the definition of this class, and Gaussian random measures $\hat{V}_{l_1, i_1}(\mathbf{x}_1, \cdot),\ldots, \hat{V}_{l_N, i_N}(\mathbf{x}_N, \cdot)$, where $\mathbf{x}_j \in \mathbb{R}^d$, $l_j \in \{0, 1\}$, $i_j \in \{1,\ldots, d\}$, $j = 1,\ldots, N$ we define the *multiple stochastic integral*, cf. ref. 23,

$$\int_{(\mathbb{R}^d)^N} \cdots \int \psi(\mathbf{k}_1,\ldots, \mathbf{k}_N)\, \hat{V}_{l_1, i_1}(\mathbf{x}_1, d\mathbf{k}_1) \cdots \hat{V}_{l_N, i_N}(\mathbf{x}_N, d\mathbf{k}_N). \tag{3.18}$$

For $\psi_1,\ldots,\psi_N \in \mathscr{S}(\mathbb{R}^d)$, the Schwartz space, we set

$$\int_{(\mathbb{R}^d)^N}\cdots\int \psi_1(\mathbf{k}_1)\cdots\psi_N(\mathbf{k}_N)\,\hat{V}_{l_1,i_1}(\mathbf{x}_1,d\mathbf{k}_1)\cdots\hat{V}_{l_N,i_N}(\mathbf{x}_N,d\mathbf{k}_N)$$
$$:=\int\psi_1(\mathbf{k}_1)\,\hat{V}_{l_1,i_1}(\mathbf{x}_1,d\mathbf{k}_1)\cdots\int\psi_N(\mathbf{k}_N)\,\hat{V}_{l_N,i_N}(\mathbf{x}_N,d\mathbf{k}_N). \tag{3.19}$$

We then extend the definition of multiple integration to the closure $\mathfrak{C}$ of the Schwartz space $\mathscr{S}((\mathbb{R}^d)^N)$ under the norm

$$\|\psi\|^2:=\int_{(\mathbb{R}^d)^N}\cdots\int\int_{(\mathbb{R}^d)^N}\cdots\int\psi(\mathbf{k}_1,\ldots,\mathbf{k}_N)\,\psi(\mathbf{k}'_1,\ldots,\mathbf{k}'_N)$$
$$\times\mathbb{E}[\hat{V}_{l_1,i_1}(\mathbf{x}_1,d\mathbf{k}_1)\cdots\hat{V}_{l_N,i_N}(\mathbf{x}_N,d\mathbf{k}_N)\,\hat{V}_{l_1,i_1}(\mathbf{x}_1,d\mathbf{k}'_1)\cdots\hat{V}_{l_N,i_N}(\mathbf{x}_N,d\mathbf{k}'_N)] \tag{3.20}$$

for any $\psi\in\mathscr{S}((\mathbb{R}^d)^N)$. The expectation is to be calculated using the following formal rule: we treat $(\hat{V}_{l_1,i_1}(\mathbf{x}_1,d\mathbf{k}_1),\ldots,\hat{V}_{l_N,i_N}(\mathbf{x}_N,d\mathbf{k}_N))$ formally as joint zero mean normal whose covariance matrix is to be calculated with the help of the following relations

$$\mathbb{E}[\hat{V}_{l,i}(\mathbf{x},d\mathbf{k})\,\hat{V}_{l',i'}(\mathbf{x}',d\mathbf{k}')]$$
$$=\delta_{l,l'}c_0(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'))\,\frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}}\left(\delta_{i,i'}-\frac{k_ik'_i}{|\mathbf{k}|^2}\right)\delta(\mathbf{k}-\mathbf{k}')\,d\mathbf{k}\,d\mathbf{k}'.$$

Let us denote $\mathbf{i}:=(i_1,\ldots,i_d)$, $\mathbf{l}:=(l_1,\ldots,l_d)$ and let $\Psi_{\mathbf{l},\mathbf{i}}$ denote the right hand side of (3.18). Note that $\Psi_{\mathbf{l},\mathbf{i}}\in\mathscr{P}^N(\mathbf{V})$—the Hilbert space obtained as a completion of the space of Nth degree polynomials in variables $\int\psi(\mathbf{k})\,\hat{\mathbf{V}}(\mathbf{x},d\mathbf{k})$ with respect to the standard L^2 norm.

Denote by $\mathscr{P}^N_{\mathrm{reg}}$ the space of all $\Psi_{\mathbf{l},\mathbf{i}}\in\mathscr{P}^N$ given by (3.18) with $\psi\in\mathscr{S}((\mathbb{R}^d)^N)$ such that $\mathbf{0}\notin\operatorname{supp}\hat{\psi}$. The following result can be obtained by a direct calculation.

Proposition 2. For any $\Psi_{\mathbf{l},\mathbf{i}}\in\mathscr{P}^N_{\mathrm{reg}}$ we have

$$\nabla\Psi_{\mathbf{l},\mathbf{i}}=\sum_{p=1}^N(-1)^{l_p}\,\mathbf{k}_p\int_{(\mathbb{R}^d)^N}\cdots\int\psi(\mathbf{k}_1,\ldots,\mathbf{k}_N)$$
$$\times\hat{V}_{l_1,i_1}(\mathbf{x}_1,d\mathbf{k}_1)\cdots\hat{V}_{l'_p,i_p}(\mathbf{x}_p,d\mathbf{k}_p)\cdots\hat{V}_{l_N,i_N}(\mathbf{x}_N,d\mathbf{k}_N). \tag{3.21}$$

Here $l'_p:=1-l_p$.

4. THE PROOF OF THEOREM 1

4.1. The Lagrangian Process Corresponding to the Passive Tracer Motion

Throughout this and the following section we shall assume that $\kappa = 1$. Suppose that the process $\eta_t := \tau_{\mathbf{x}(t)}(\omega)$, where $\mathbf{x}(t)$, $t \geqslant 0$ is given by (1.1). It is well known that this process is Markovian, see, e.g., ref. 18, p. 40, and thanks to the incompressibility assumption μ is invariant, see ibid. p. 41. We also have

$$\mathbf{x}(t) \cdot \mathbf{e} = \int_0^t V_{\mathbf{e}}(\eta_s)\, ds + \sqrt{2}\, \mathbf{w}(t) \cdot \mathbf{e}, \tag{4.1}$$

where $V_{\mathbf{e}}(\omega) := \mathbf{V}(\omega) \cdot \mathbf{e}$, see (3.14).

The following proposition holds, see ref. 14, Lemma 4.1.

Proposition 3. $\mathscr{P}_{\text{reg}}$ is a core of the L^2-generator $\mathscr{L}$ of the process $(\eta_t)_{t \geqslant 0}$. Moreover

$$\mathscr{L}\Phi = \Delta\Phi + \mathbf{V} \cdot \nabla\Phi, \qquad \Phi \in \mathscr{P}_{\text{reg}}. \tag{4.2}$$

Moreover $\mathscr{P}_{\text{reg}} \subseteq D(S)$ (the domain of the symmetric part of the generator). The symmetric and anti-symmetric parts are correspondingly equal to

$$S\Phi = \Delta\Phi, \qquad A\Phi = \mathbf{V} \cdot \nabla\Phi, \qquad \Phi \in \mathscr{P}_{\text{reg}}.$$

4.2. The Proof of (2.3)

Denote by $\mathscr{P}^1_{\text{reg},\,0}$ the space of all degree one, regular polynomials of zero mean. Suppose that

$$\Psi := \int_{\mathbb{R}^d} \psi(\mathbf{k}) \cdot \hat{\mathbf{V}}_0(\mathbf{0}, d\mathbf{k}) \in \mathscr{P}^1_{\text{reg},\,0}. \tag{4.3}$$

As the first step towards establishing (2.5) we calculate the supremum $d_*(t)$ of $\mathscr{S}(\Psi)$, cf. (3.6), over all Ψ of the form (4.3) with $V = V_{\mathbf{e}}$.

Lemma 3. We have

$$d_*(t) = \int_{\mathbb{R}^d} \frac{(\Gamma(\mathbf{k})\, \mathbf{e}, \mathbf{e})_{\mathbb{R}^d}\, a(|\mathbf{k}|)\, d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\, \mathscr{H}(t, \mathbf{k})}, \tag{4.4}$$

where

$$\mathscr{H}(t, \mathbf{k}) := \frac{1}{t} + |\mathbf{k}|^2 + \int_{\mathbb{R}^d} \frac{1}{\frac{1}{t} + |\mathbf{k} - \mathbf{k}'|^2} \times \frac{(\Gamma(\mathbf{k}')\, \mathbf{k}, \mathbf{k})_{\mathbb{R}^d}\, a(|\mathbf{k}'|)\, d\mathbf{k}'}{|\mathbf{k}'|^{2\alpha+d-2}}. \tag{4.5}$$

Proof. A direct calculation shows that for any Ψ of the form (4.3) we have

$$2(V_{\mathbf{e}}, \Psi)_{L^2} = 2\int_{\mathbb{R}^d} \frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}), \mathbf{e})_{\mathbb{R}^d}\, a(|\mathbf{k}|)\, d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}}. \tag{4.6}$$

Using (3.21) we calculate

$$\begin{aligned}\|\Psi\|_{1,1/t}^2 &= ((1/t-\Delta)\,\Psi, \Psi)_{L^2}\\ &= \int_{\mathbb{R}^d}\left(\frac{1}{t}+|\mathbf{k}|^2\right)\frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}), \psi(\mathbf{k}))_{\mathbb{R}^d}\, a(|\mathbf{k}|)\, d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}}.\end{aligned} \tag{4.7}$$

Applying (3.21) once more we obtain

$$A\Psi = \mathbf{V}\cdot\nabla\Psi = \sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi_i(\mathbf{k})\,k_j\hat{V}_{1,i}(d\mathbf{k})\,\hat{V}_{0,j}(d\mathbf{k}').$$

Here we used the abbreviated notation $\hat{\mathbf{V}}_i(d\mathbf{k}) = (\hat{V}_{i,1}(d\mathbf{k}),\ldots, \hat{V}_{i,d}(d\mathbf{k}))$ for $i = 0, 1$.

Note that

$$\|A\Psi\|_{-1,1/t}^2 = (A\Psi, \Phi)_{L^2}, \tag{4.8}$$

where Φ is the unique solution of the equation

$$(1/t-\Delta)\,\Phi = A\Psi. \tag{4.9}$$

We look for the solutions of (4.9) among the polynomials belonging to $\mathscr{P}^2_{\mathrm{reg}}$ of the form

$$\begin{aligned}\Phi = &\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\alpha_{i,j}(\mathbf{k},\mathbf{k}')\,\hat{V}_{1,i}(d\mathbf{k})\,\hat{V}_{0,j}(d\mathbf{k}')\\ &+\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\beta_{i,j}(\mathbf{k},\mathbf{k}')\,\hat{V}_{0,i}(d\mathbf{k})\,\hat{V}_{1,j}(d\mathbf{k}').\end{aligned} \tag{4.10}$$

Then, after a straightforward calculation, using (3.21), we obtain

$$\begin{aligned}-\Delta\Phi = &\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[(|\mathbf{k}|^2+|\mathbf{k}'|^2)\,\alpha_{i,j}(\mathbf{k},\mathbf{k}')\\ &+2\mathbf{k}\cdot\mathbf{k}'\beta_{i,j}(\mathbf{k},\mathbf{k}')]\,\hat{V}_{1,i}(d\mathbf{k})\,\hat{V}_{0,j}(d\mathbf{k}')\\ &+\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[2\mathbf{k}\cdot\mathbf{k}'\alpha_{i,j}(\mathbf{k},\mathbf{k}')+(|\mathbf{k}|^2+|\mathbf{k}'|^2)\,\beta_{i,j}(\mathbf{k},\mathbf{k}')]\\ &\times\hat{V}_{0,i}(d\mathbf{k})\,\hat{V}_{1,j}(d\mathbf{k}').\end{aligned} \tag{4.11}$$

Thus, substituting into (4.9), we get the following system of equations

$$\begin{cases} (\frac{1}{t}+|\mathbf{k}|^2+|\mathbf{k}'|^2)\,\alpha_{i,j}(\mathbf{k},\mathbf{k}')+2\mathbf{k}\cdot\mathbf{k}'\beta_{i,j}(\mathbf{k},\mathbf{k}')=\psi_i(\mathbf{k})\,k_j\\ 2\mathbf{k}\cdot\mathbf{k}'\alpha_{i,j}(\mathbf{k},\mathbf{k}')+(\frac{1}{t}+|\mathbf{k}|^2+|\mathbf{k}'|^2)\,\beta_{i,j}(\mathbf{k},\mathbf{k}')=0.\end{cases}$$

Solving it, we obtain

$$\begin{cases} \alpha_{i,j}(\mathbf{k},\mathbf{k}')=\dfrac{1}{2}\left(\dfrac{\psi_i(\mathbf{k})\,k_j}{\frac{1}{t}+|\mathbf{k}-\mathbf{k}'|^2}+\dfrac{\psi_i(\mathbf{k})\,k_j}{\frac{1}{t}+|\mathbf{k}+\mathbf{k}'|^2}\right)\\ \beta_{i,j}(\mathbf{k},\mathbf{k}')=\dfrac{1}{2}\left(\dfrac{\psi_i(\mathbf{k})\,k_j}{\frac{1}{t}+|\mathbf{k}-\mathbf{k}'|^2}-\dfrac{\psi_i(\mathbf{k})\,k_j}{\frac{1}{t}+|\mathbf{k}+\mathbf{k}'|^2}\right).\end{cases} \tag{4.12}$$

Substituting for $\alpha_{i,j}$, $\beta_{i,j}$ into (4.10) and then subsequently using (4.8) we conclude that

$$\|A\Psi\|^2_{-1,1/t}=\mathscr{A}(\psi),$$

where

$$\mathscr{A}(\psi):=\frac{1}{2}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}),\psi(\mathbf{k}))_{\mathbb{R}^d}\,(\Gamma(\mathbf{k}')\,\mathbf{k},\mathbf{k})_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,a(|\mathbf{k}'|)\,d\mathbf{k}\,d\mathbf{k}'}{|\mathbf{k}|^{2\alpha+d-2}\,|\mathbf{k}'|^{2\alpha+d-2}\,(\frac{1}{t}+|\mathbf{k}-\mathbf{k}'|^2)}\right.$$
$$\left.+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}),\psi(\mathbf{k}))_{\mathbb{R}^d}\,(\Gamma(\mathbf{k}')\,\mathbf{k},\mathbf{k})_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,a(|\mathbf{k}'|)\,d\mathbf{k}\,d\mathbf{k}'}{|\mathbf{k}|^{2\alpha+d-2}\,|\mathbf{k}'|^{2\alpha+d-2}\,(\frac{1}{t}+|\mathbf{k}+\mathbf{k}'|^2)}\right].$$

Summarizing, we have shown that

$$d_*(t)=\sup_{\psi\in\mathscr{S}(\mathbb{R}^d;\,\mathbb{R}^d)}\mathscr{G}(\psi),$$

where

$$\mathscr{G}(\psi):=2\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}),\mathbf{e})_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}}$$
$$-\int_{\mathbb{R}^d}\left(\frac{1}{t}+|\mathbf{k}|^2\right)\frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}),\psi(\mathbf{k}))_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}}-\mathscr{A}(\psi).$$

Using a standard variational calculus one concludes that the maximizer of $\mathscr{G}(\cdot)$ is given by

$$\psi_0=\mathscr{H}^{-1}(t,\mathbf{k})\,\mathbf{e},$$

with $\mathscr{H}(\cdot,\cdot)$ given by (4.5). In addition, the corresponding maximal value of $\mathscr{G}(\cdot)$ equals

$$\mathscr{G}(\psi_0)=\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\psi_0(\mathbf{k}),\mathbf{e})_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}}$$

and the conclusion of the lemma follows. ∎

Substituting $\mathbf{k} := \sqrt{t}\,\mathbf{k}$, $\mathbf{k}' := \sqrt{t}\,\mathbf{k}'$ in the integrals appearing on the right hand side of (4.4) we conclude that

$$d_*(t) = t^\alpha \int_{\mathbb{R}^d} \frac{(\Gamma(\mathbf{k})\,\mathbf{e},\mathbf{e})_{\mathbb{R}^d}\, a(t^{-1/2}\,|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\,\mathscr{H}'(t,\mathbf{k})}, \tag{4.13}$$

where

$$\begin{aligned}\mathscr{H}'(t,\mathbf{k}) &:= 1+|\mathbf{k}|^2+t^\alpha \int_{\mathbb{R}^d} \frac{1}{1+|\mathbf{k}-\mathbf{k}'|^2} \times \frac{(\Gamma(\mathbf{k}')\,\mathbf{k},\mathbf{k})_{\mathbb{R}^d}\, a(|\mathbf{k}'|/\sqrt{t})\,d\mathbf{k}'}{|\mathbf{k}'|^{2\alpha+d-2}}\\ &\leqslant 1+|\mathbf{k}|^2+t^\alpha\,|\mathbf{k}|^2\,\mathscr{H}_0(t,|\mathbf{k}|),\end{aligned}$$

where

$$\mathscr{H}_0(t,k) := \|a\|_{L^\infty} \int_0^{K\sqrt{t}} \frac{du}{u^{2\alpha-1}[1+(u-k)^2]}. \tag{4.14}$$

Consider first the case $\alpha > 1/2$. We can estimate then the integral appearing on the right hand side of (4.14) by

$$\frac{C}{1+k^2}\int_0^{k/2}\frac{du}{u^{2\alpha-1}} + \frac{C}{k^{2\alpha-1}}\int_{k/2}^{+\infty}\frac{du}{1+u^2} \leqslant \frac{C}{k^{2\alpha-1}}.$$

Hence,

$$d_*(t) \geqslant Ct^\alpha \int_{\mathbb{R}^d} \frac{(\Gamma(\mathbf{k})\,\mathbf{e},\mathbf{e})_{\mathbb{R}^d}\, a(t^{-1/2}\,|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\,(1+|\mathbf{k}|^2+|\mathbf{k}|^{3-2\alpha}\,t^\alpha)}.$$

After a substitution $\mathbf{k} := t^{\alpha/(3-2\alpha)}\mathbf{k}$ we obtain

$$d_*(t) \geqslant C_1 t^{\alpha/(3-2\alpha)}$$

for some positive constant $C_1 > 0$ and (2.4) follows.

In case when $\alpha \in (0, 1/2)$ we have

$$\mathscr{H}_0(t,k) \leqslant Ck^{1-2\alpha}\int_0^{2k}\frac{du}{1+(k-u)^2} + C\int_{2k}^{+\infty}\frac{du}{u^{2\alpha-1}(1+u^2)} \leqslant C_1(k^{1-2\alpha}+1).$$

Finally, we arrive at the following estimate

$$d_*(t) \geqslant c\,t^\alpha \int_{\mathbb{R}^d} \frac{(\Gamma(\mathbf{k})\,\mathbf{e},\mathbf{e})_{\mathbb{R}^d}\, a(t^{-1/2}\,|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\,(1+|\mathbf{k}|^2+t^\alpha\,|\mathbf{k}|^2+t^\alpha\,|\mathbf{k}|^{3-2\alpha})}.$$

After the substitution $\mathbf{k} := t^{\alpha/2}\mathbf{k}$ we get

$$d_*(t) \geqslant ct^{\alpha^2} \int_{\mathbb{R}^d} \frac{(\Gamma(\mathbf{k})\, \mathbf{e}, \mathbf{e})_{\mathbb{R}^d}\, a(t^{-(1+\alpha)/2}\, |\mathbf{k}|)\, d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\, [1+|\mathbf{k}|^2\,(1+t^{-\alpha})+t^{\alpha(\alpha-1/2)}\, |\mathbf{k}|^{3-2\alpha}]}$$

from which (2.3) follows. ▌

4.3. The Proof of (2.5)

We use the variational principle expressed by (3.7) and the estimate (3.8) to obtain the upper bound. We take $F=0$ as a test function in (3.7). Then,

$$\|V_{\mathbf{e}}\|^2_{-1,\,1/t} = \int_{\mathbb{R}^d} \frac{\gamma_{\mathbf{e}}(\mathbf{k})\, a(|\mathbf{k}|)\, d\mathbf{k}}{(|\mathbf{k}|^2+\frac{1}{t})\, |\mathbf{k}|^{2\alpha+d-2}} \leqslant C \int_0^K \frac{dk}{(k^2+\frac{1}{t})\, k^{2\alpha-1}}. \tag{4.15}$$

Substituting $k := t^{1/2}k$ we conclude that the utmost right hand side of (4.15) is of order of magnitude

$$Ct^{\alpha} \int_0^{+\infty} \frac{dk}{(k^2+1)\, k^{2\alpha-1}} \qquad \text{for} \quad t \gg 1$$

and (2.3) follows. ▌

5. THE PROOF OF THEOREM 2

In this case the environment is non-static and can be described as the time stationary solution of an infinite dimensional linear stochastic differential equation

$$d\omega(t) = -A\omega(t)\, dt + B\, dW(t). \tag{5.1}$$

$(\omega(t))_{t\geqslant 0}$ takes values in Ω, see ref. 11. Here $W(\cdot)$ is a cylindrical Wiener process on $L^2_{\mathrm{div}}(\mathbb{R}^d, \mathbb{R}^d)$—the space of all square integrable, incompressible d-dimensional vector fields—defined over the probability space $\mathcal{T}_2$. B: $L^2_{\mathrm{div}}(\mathbb{R}^d, \mathbb{R}^d) \to \Omega$ is the continuous extension of

$$\widehat{B\psi}\,(\mathbf{k}) = \sqrt{2\mathscr{E}(|\mathbf{k}|)}\; |\mathbf{k}|^{(1+2\beta-d)/2}\, \hat{\psi}(\mathbf{k}), \qquad \psi \in \mathscr{C}^{\infty}_{0,\,\mathrm{div}}, \tag{5.2}$$

where $\beta \geqslant 0$. It can be shown, see part (1) of Proposition 2 of ref. 11, that B is a Hilbert–Schmidt operator.

A, on the other hand, is the generator of a semigroup $S(\cdot)$ given by

$$\widehat{S(t)\,\psi}\,(\mathbf{k}) := \mathrm{e}^{-|\mathbf{k}|^{2\beta}\, t}\, \hat{\psi}(\mathbf{k}), \qquad \psi \in \mathscr{C}^{\infty}_{0,\,\mathrm{div}}. \tag{5.3}$$

It can be shown, see part (2) of Proposition 2 ibid., that $(S(t))_{t\geqslant 0}$ extends to a C_0-semigroup of operators on Ω, see, e.g., ref. 11, provided that $\rho\in(d/2, d/2+\beta)$.

We can write that the particle displacement is given by (4.1), with $\eta_t := \tau_{\mathbf{x}(t)}(\omega(t))$, $t\geqslant 0$. We formulate an analogue of Proposition 3, cf. Lemma 6.1 of ref. 14.

Proposition 4. $\mathscr{P}_{\mathrm{reg}}$ is a core of the L^2-generator $\mathscr{L}$ of the process η_t, $t\geqslant 0$. Moreover,

$$\mathscr{L}F = \Delta F + LF + \mathbf{V}\cdot\nabla F, \qquad F\in\mathscr{P}_{\mathrm{reg}}. \tag{5.4}$$

Here L is the L^2-generator of $\omega(\cdot)$. Moreover $\mathscr{P}_{\mathrm{reg}}\subseteq\mathscr{D}(S)$, the domain of the symmetric part of the generator, and the symmetric and anti-symmetric parts are correspondingly $SF=\Delta F+LF$, $AF=\mathbf{V}\cdot\nabla F$, $F\in\mathscr{P}_{\mathrm{reg}}$.

By a straightforward direct calculation one can establish the following.

Proposition 5. We have $\mathscr{P}^N_{\mathrm{reg}}\subseteq D(L)$. In addition, for any $\mathbf{l}=(l_1,\ldots, l_N)\in\{0, 1\}^N$, $\mathbf{i}=(i_1,\ldots, i_N)\in\{1,\ldots, d\}^N$ and $\Psi_{\mathbf{l},\mathbf{i}}$ given by (3.18) we have

$$\begin{aligned} L\Psi_{\mathbf{l},\mathbf{i}} = &-\int\cdots\int_{(\mathbb{R}^d)^N} (|\mathbf{k}_1|^{2\beta}+\cdots+|\mathbf{k}_N|^{2\beta})\,\psi(\mathbf{k}_1,\ldots,\mathbf{k}_N)\\ &\times\hat{V}_{l_1,i_1}(\mathbf{x}_1, d\mathbf{k}_1)\cdots\hat{V}_{l_N,i_N}(\mathbf{x}_N, d\mathbf{k}_N). \end{aligned} \tag{5.5}$$

The proof of Theorem 2 follows the same line as the corresponding argument for Theorem 1 and relies on the use of variational principles expressed in Proposition 1 together with the accompanying Lemma 1.

5.1. The Proof of the Lower Bounds on γ_*

Note that for Ψ given by (4.3) we have, after the calculations similar to those performed in Section 4.2,

$$\begin{aligned} \|\Psi\|_{1,1/t} &= \left(\left(\frac{1}{t}-L-\Delta\right)\Psi,\Psi\right)_{L^2}\\ &= \int_{\mathbb{R}^d}\left(\frac{1}{t}+|\mathbf{k}|^{2\beta}+|\mathbf{k}|^2\right)\frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}),\psi(\mathbf{k}))_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}} \end{aligned}$$

and

$$\|A\Psi\|_{-1,\,1/t}$$
$$=\frac{1}{2}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}),\psi(\mathbf{k}))_{\mathbb{R}^d}\,(\Gamma(\mathbf{k}')\,\mathbf{k},\mathbf{k})_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,a(|\mathbf{k}'|)\,d\mathbf{k}\,d\mathbf{k}'}{|\mathbf{k}|^{2\alpha+d-2}\,|\mathbf{k}'|^{2\alpha+d-2}\,(\frac{1}{t}+|\mathbf{k}|^{2\beta}+|\mathbf{k}'|^{2\beta}+|\mathbf{k}-\mathbf{k}'|^2)}\right.$$
$$\left.+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\psi(\mathbf{k}),\psi(\mathbf{k}))_{\mathbb{R}^d}\,(\Gamma(\mathbf{k}')\,\mathbf{k},\mathbf{k})_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,a(|\mathbf{k}'|)\,d\mathbf{k}\,d\mathbf{k}'}{|\mathbf{k}|^{2\alpha+d-2}\,|\mathbf{k}'|^{2\alpha+d-2}\,(\frac{1}{t}+|\mathbf{k}|^{2\beta}+|\mathbf{k}'|^{2\beta}+|\mathbf{k}+\mathbf{k}'|^2)}\right].$$

The supremum $d_*(t)$ of $\mathscr{S}(\cdot)$, cf. (3.6), over the test functions of the form (4.3) can be calculated exactly as in Lemma 3, which leads to

$$d_*(t)=\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\mathbf{e},\mathbf{e})_{\mathbb{R}^d}\,a(|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\,\mathscr{H}_\beta(t,\mathbf{k})},\tag{5.6}$$

where

$$\mathscr{H}_\beta(t,\mathbf{k}):=\frac{1}{t}+|\mathbf{k}|^2+|\mathbf{k}|^{2\beta}$$
$$+\int_{\mathbb{R}^d}\frac{1}{\frac{1}{t}+|\mathbf{k}|^{2\beta}+|\mathbf{k}'|^{2\beta}+|\mathbf{k}-\mathbf{k}'|^2}\times\frac{(\Gamma(\mathbf{k}')\,\mathbf{k},\mathbf{k})_{\mathbb{R}^d}\,a(|\mathbf{k}'|)\,d\mathbf{k}'}{|\mathbf{k}'|^{2\alpha+d-2}}\tag{5.7}$$

5.1.1. *The Case (i)*

We substitute $\mathbf{k}:=t^{1/(2\beta)}\mathbf{k}$, $\mathbf{k}':=t^{1/(2\beta)}\mathbf{k}'$ in (5.6) and (5.7). Then,

$$d_*(t)=t^{(\alpha+\beta-1)/\beta}\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\mathbf{e},\mathbf{e})_{\mathbb{R}^d}\,a(|\mathbf{k}|\,t^{-1/(2\beta)})\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\,\mathscr{H}'_\beta(t,\mathbf{k})},\tag{5.8}$$

where

$$\mathscr{H}'_\beta(t,\mathbf{k}):=1+t^{1-1/\beta}\,|\mathbf{k}|^2+|\mathbf{k}|^{2\beta}$$
$$+t^{(\alpha+2\beta-2)/\beta}\int_{\mathbb{R}^d}\frac{1}{1+|\mathbf{k}|^{2\beta}+|\mathbf{k}'|^{2\beta}+t^{1-1/\beta}\,|\mathbf{k}-\mathbf{k}'|^2}$$
$$\times\frac{(\Gamma(\mathbf{k}')\,\mathbf{k},\mathbf{k})_{\mathbb{R}^d}\,a(|\mathbf{k}'|\,t^{-1/(2\beta)})\,d\mathbf{k}'}{|\mathbf{k}'|^{2\alpha+d-2}}.\tag{5.9}$$

When $\alpha+\beta>1$ and $\alpha+2\beta<2$ (note that then necessarily $\beta<1$, see Fig. 1), we have, by virtue of the Lebesgue Dominated Convergence Theorem

$$d_*(t)\geqslant c_1 t^{(\alpha+\beta-1)/\beta},\qquad \forall t>0\tag{5.10}$$

for some $c_1>0$.

5.1.2. The Case (ii)

$\alpha+2\beta>2$ and $\beta<1$. We use the same substitution as in the previous section and estimate

$$\mathcal{H}'_\beta(t,\mathbf{k})\leqslant 1+t^{1-1/\beta}\,|\mathbf{k}|^2+|\mathbf{k}|^{2\beta}+t^{(\alpha+2\beta-2)/\beta}\,|\mathbf{k}|^2\int_0^{Kt^{1/(2\beta)}}\frac{dk'}{(k')^{2\alpha-1}\,(1+|\mathbf{k}|^{2\beta}+(k')^{2\beta})}. \tag{5.11}$$

After the substitution $u:=k'(1+|\mathbf{k}|^{2\beta})^{-1/(2\beta)}$ in the integral on the right hand side of (5.11) we obtain the following upper bound on the left hand side of (5.11)

$$\begin{aligned}&1+t^{1-1/\beta}\,|\mathbf{k}|^2+|\mathbf{k}|^{2\beta}+\mathcal{I}_0 t^{(\alpha+2\beta-2)/\beta}\,|\mathbf{k}|^2\,(1+|\mathbf{k}|^{2\beta})^{(1-\alpha-\beta)/(2\beta)}\\&\quad\leqslant 1+t^{1-1/\beta}\,|\mathbf{k}|^2+|\mathbf{k}|^{2\beta}+\mathcal{I}_0 t^{(\alpha+2\beta-2)/\beta}\,|\mathbf{k}|^{2(2-\alpha-\beta)},\end{aligned}$$

with

$$\mathcal{I}_0:=\int_0^{+\infty}\frac{du}{u^{2\alpha-1}(1+u^{2\beta})}<+\infty,\qquad \text{for}\quad \alpha+\beta>1.$$

Therefore, by virtue of (5.8), we can estimate

$$d_*(t)\geqslant t^{(\alpha+\beta-1)/\beta}\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\mathbf{e},\mathbf{e})_{\mathbb{R}^d}\,a(|\mathbf{k}|\,t^{-1/(2\beta)})\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\,(1+t^{1-1/\beta}\,|\mathbf{k}|^2+|\mathbf{k}|^{2\beta}+\mathcal{I}_0 t^{(\alpha+2\beta-2)/\beta}\,|\mathbf{k}|^{2(2-\alpha-\beta)})}.$$

After performing the substitution $\mathbf{l}:=t^{(\alpha+2\beta-2)/[2\beta(2-\alpha-\beta)]}\mathbf{k}$ we conclude that

$$\liminf_{t\uparrow+\infty}\frac{d_*(t)}{t^{(1-\beta)/(2-\alpha-\beta)}}\geqslant C,$$

with

$$C:=a(0)\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{l})\,\mathbf{e},\mathbf{e})_{\mathbb{R}^d}\,d\mathbf{l}}{|\mathbf{l}|^{2\alpha+d-2}\,(1+\mathcal{I}_0\,|\mathbf{l}|^{2(2-\alpha-\beta)})}.$$

5.1.3. The Case (iii)

In this case we change variables $\mathbf{k}:=\sqrt{t}\,\mathbf{k}$, $\mathbf{k}':=\sqrt{t}\,\mathbf{k}'$ and arrive at

$$d_*(t)=t^\alpha\int_{\mathbb{R}^d}\frac{(\Gamma(\mathbf{k})\,\mathbf{e},\mathbf{e})_{\mathbb{R}^d}\,a(t^{-1/2}\,|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\alpha+d-2}\,\mathcal{H}''_\beta(t,\mathbf{k})},$$

where

$$\mathscr{H}''_{\beta}(t, \mathbf{k}) := 1+|\mathbf{k}|^2+|\mathbf{k}|^{2\beta}\, t^{1-\beta}$$
$$+t^{\alpha} \int_{\mathbb{R}^d} \frac{1}{1+|\mathbf{k}|^{2\beta}\, t^{1-\beta}+|\mathbf{k}'|^{2\beta}\, t^{1-\beta}+|\mathbf{k}-\mathbf{k}'|^2}$$
$$\times \frac{(\Gamma(\mathbf{k}')\, \mathbf{k}, \mathbf{k})_{\mathbb{R}^d}\, a(|\mathbf{k}'|/\sqrt{t})\, d\mathbf{k}'}{|\mathbf{k}'|^{2\alpha+d-2}}.$$

Since $\beta \geqslant 1$ the influence of the terms $|\mathbf{k}|^{2\beta}\, t^{1-\beta}$, $|\mathbf{k}'|^{2\beta}\, t^{1-\beta}$ appearing in the above expression is negligible as $t \gg 1$. In order to obtain a lower bound on $d_*(t)$ we can repeat therefore the argument of Section 4.2 and obtain as a result the conclusion of part (iii).

5.2. The Proof of the Upper Bounds on γ^*

We follow the same route as in Section 4.3 and use the variational principle as expressed by (3.7) with $F=0$. Note that

$$\|V_{\mathbf{e}}\|^2_{-1,\, 1/t} = \int_{\mathbb{R}^d} \frac{\gamma_{\mathbf{e}}(\mathbf{k})\, a(|\mathbf{k}|)\, d\mathbf{k}}{(|\mathbf{k}|^2+|\mathbf{k}|^{2\beta}+\frac{1}{t})\, |\mathbf{k}|^{2\alpha+d-2}} \leqslant C \int_0^K \frac{dk}{(k^2+k^{2\beta}+\frac{1}{t})\, k^{2\alpha-1}}. \tag{5.12}$$

We consider two cases. First, when $\beta<1$, then we substitute $k := t^{1/(2\beta)}k$ and obtain the upper bounds as claimed in parts (i) and (ii) of Theorem 2.

When, on the other hand $\beta>1$ we conclude the claim (iii) by substituting $k := t^{1/2}k$ in the integral expression on the right hand side of (5.12). ∎

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